JOURNAL OF COMPUTATIONAL PHYSICS 1, 471-484 (1967)

Second-Order Numerical Advection

W. P. CROWLEY

Lawrence Radiation Laboratory, University of California
Livermore, California

ABSTRACT

A second order accurate numerical approximation to the advection terms in the primitive equations of hydrodynamics is proposed. The scheme is compared with a similar second-order scheme and a first-order scheme in view of aliasing effects and of amplitude and phase errors. A stability criterion is derived for the one-dimensional case and one is suggested for the two-dimensional case. Numerical results from simple test calculations are given.

INTRODUCTION

This paper describes a second-order numerical approximation to the advective terms in the time-dependent nonlinear equations of motion associated with long-term atmospheric and oceanic hydrodynamics, the so-called primitive equations. The method is applied first to a one-space-dimensional equation, and the amplitude damping introduced by the finite-difference approximations is compared with a similar second-order scheme and with a first–order scheme. The relative merits of the three schemes in view of aliasing and of the phase error introduced by the finite-difference approximations are also discussed.

The method is then applied to advection in two space dimensions, and a condition necessary for computational stability is suggested.

Finally, results of simple one-dimensional test calculations involving advection of a Gaussian distribution are presented.

Second-order accuracy in numerical hydrodynamic calculations is achieved quite easily by schemes which employ centered time differences. Unfortunately, centered-time-difference schemes require complete knowledge of the state vector for times t and $t - \Delta t$ in order to advance the system to $t + \Delta t$, and thus have computer-memory requirements which are approximately double those of two time-step schemes. They also have a disadvantage in that they produce two divergent solutions at adjacent time steps [1].

It is also possible to attain second-order accuracy with one-sided time differences as Lax and Wendroff [2] have indicated, but their method involves quite complicated spatial derivatives which necessitate a large number of arithmetic operations per mesh point. An alternative plan is to use a two-step version of the Lax–Wendroff scheme as proposed by Richtmyer [3]. Both of these methods have been used with success by Burstein [4] in compressible hydrodynamic calculations.

The proposed scheme is based on a two-time-step scheme—i.e., the state vector is known at time t, and with this information and some intermediate calculated information at $t + \frac{1}{2}(\Delta t)$, the system is advanced to time $t + \Delta t$. In this way, this scheme is similar to the Lax–Wendroff–Ritchmyer two-step scheme, but there are two immediate differences. The equations which comprise this scheme are not required to be in conservation form as they are in Lax–Wendroff, and since, in this scheme every space point is calculated each time cycle, the two out-of-phase solutions which appear at adjacent mesh points in Lax–Wendroff [5] cannot appear here.

One-Dimensional Formulation

Consider

$$\psi_t + u\psi_x = S(x, t, \psi, \psi_x, \psi_{xx}), \tag{1}$$

where $\psi = \psi(x, t)$ and $u = u(x, t)$. For simplicity $u(x, t)$ is assumed known, but, in general, it is one of the elements in the vector ψ. For example, if $\psi = u$ and $S = \nu u_{xx}$, Eq. (1) becomes Burgers' equation, a one-dimensional version of the Navier–Stokes equation with acceleration due to pressure gradients, Coriolis effects, etc., eliminated.

In the following analysis, S will be taken to be zero, but its inclusion in the finite-difference approximations can be accomplished in a straightforward manner.

Expanding $\psi(x, t)$ in a Taylor series, and denoting $\psi(x, t) = \psi(j\Delta x, N\Delta t)$ by ψ_j^N,

$$\psi_j^{N+1} = \psi_j^N + \frac{\partial \psi}{\partial t}\bigg|_j^N \Delta t + \frac{\partial^2 \psi}{\partial t^2}\bigg|_j^N \frac{\Delta t^2}{2} + O(\Delta t^3). \tag{2}$$

If the first three terms in the expansion are available, then ψ is said to be second-order accurate, or the error in amplitude of ψ due to truncation is third order.

Equation (2) may be written as

$$\psi_j^{N+1} = \psi_j^N + \frac{\partial}{\partial t}\left\{\psi + \frac{\partial \psi}{\partial t}\frac{\Delta t}{2}\right\}\bigg|_j^N \Delta t + O(\Delta t^3). \tag{3}$$

Since the bracketed term (evaluated at j and N) is $\psi_j^{N+1/2}$ to first order, the expansion may also be written

$$\psi_j^{N+1} = \psi_j^N + \frac{\partial \psi}{\partial t}\Big|_j^{N+1/2} \Delta t + O(\Delta t^3), \tag{4}$$

where

$$\psi_j^{N+1/2} = \psi_j^N + \frac{\partial \psi}{\partial t}\Big|_j^N \frac{\Delta t}{2} + O(\Delta t^2). \tag{5}$$

Thus it appears that a two-stage process will give the desired results. That is, ψ is first approximated at $N + \frac{1}{2}$ by a first order scheme, and then using these intermediate and less accurate results, ψ_j^{N+1} may be obtained to second order.

Using (1) to evaluate $\partial\psi/\partial t$,

$$\frac{\partial \psi}{\partial t}\Big|_j^N = -u \frac{\partial \psi}{\partial x}\Big|_j^N \tag{6}$$

$$= -\frac{u}{2\Delta x}(\psi_{j+1}^N - \psi_{j-1}^N) + O(\Delta x^2). \tag{7}$$

The combination of (7) with (5) is first-order accurate, but unconditionally unstable. Fortunately this first step in the calculation only has to be first-order accurate, so that a higher-order term may be added which will stabilize this step and not decrease the accuracy of the final step.

Let two finite-difference operators $\bar{A}$ and $\bar{C}$ be defined by

$$\bar{A}\psi_j = \frac{\alpha_j}{4}(\psi_{j+1} - \psi_{j-1}) - \frac{1}{2}\left(\frac{\alpha_j}{2}\right)^2 (\psi_{j+1} - 2\psi_j + \psi_{j-1}), \tag{8}$$

$$\bar{C}\psi_j = \frac{\alpha_j}{2}(\psi_{j+1} - \psi_{j-1}), \tag{9}$$

where

$$\alpha_j = u_j \Delta t/\Delta x \qquad \text{and} \qquad \bar{A}\psi_j = (\bar{A}\psi)_j .$$

Then (5) becomes, when the time derivative is replaced by a space derivative plus a higher-order term that introduces stability,

$$\psi_j^{N+1/2} = (I - \bar{A})\psi_j^N. \tag{10}$$

In (4), the time derivative may be replaced by a space derivative via (1) and (7). Thus

$$\psi_j^{N+1} = \psi_j^N - \bar{C}\psi_j^{N+1/2}. \tag{11}$$

Incorporating (10) into (11), finally

$$\psi_j^{N+1} = (I - \bar{C}(I - \bar{A}))\,\psi_j^N. \tag{12}$$

Numerical stability requires that the eigenvalues of the operator $(I - \bar{C} + \bar{C}\bar{A})$ be on or inside the unit circle. If $\psi_j = \epsilon^{ikj\Delta x}$, i.e., if an eigenvector is substituted for ψ, it is found that the square of the magnitude of the eigenvalues of the operator in (12) are

$$|\,\xi_k\,|^2 = 1 - \left(1 - \frac{\alpha^2}{4}\right)\left(\frac{\alpha^4}{4}\sin^2\theta\right)(1 - \cos\theta)^2 \tag{13}$$

where $\theta = k\Delta x$.

Thus, conditional stability is attained if $|\,u\,|\,\Delta t/\Delta x < 2$ and amplitude damping goes as α^4.

A scheme proposed by Richtmyer [3] (known as the Two-Step Lax–Wendroff Scheme) differs from this proposed scheme in that the first step is

$$\psi_j^{N+1/2} = \tfrac{1}{2}(\psi_{j+1}^N + \psi_{j-1}^N) - \frac{\alpha_j}{4}(\psi_{j+1}^N - \psi_{j-1}^N), \tag{14}$$

rather than Eq. (10). Stability analysis of Richtmyer's scheme shows that the eigenvalues are

$$|\,\xi_k\,|^2 = 1 - \alpha^2\left(1 - \frac{\alpha^2}{4}\right)\sin^4\theta \tag{15}$$

so that numerical stability requirements are the same, but the damping goes as α^2 rather than α^4.

A typical first-order-accurate scheme (first-order in time, second-order in space) such as that used by Leith [6] and mentioned by Richtmyer [3] is

$$\psi_j^{N+1} = \psi_j^N - \frac{\alpha_j}{2}(\psi_{j+1}^N - \psi_{j-1}^N) + \tfrac{1}{2}\alpha_j^2(\psi_{j+1}^N - 2\psi_j^N + \psi_{j-1}^N). \tag{16}$$

This scheme has eigenvalues $|\,\xi_k\,|^2 = 1 - \alpha^2(1 - \alpha^2)(1 - \cos\theta)^2$ so that the condition for stability is more stringent, namely $|\,\alpha\,| < 1$, and the amplitude damping goes as α^2.

Aliasing

Calculations by finite difference methods have a limitation in the range of wave numbers that can be represented. For a mesh of M points, the maximum representable wave number is $\frac{1}{2}M$.

If the ψ_j are expanded in terms of the eigenfunctions $\epsilon^{ipj\Delta x}$ and if an analysis similar to that used by Phillips [7] is applied to these difference equations, it can

be shown that the advection terms cause modes p and q to interact to change the energy in modes $p + q$ and $p - q$. Since both p and q can be any integer between $-\frac{1}{2}M$ and $\frac{1}{2}M$, the interaction $p + q$ can produce a mode which is outside the range of representable wave numbers. The energy, rather than disappearing, is falsified, or aliased into modes $(p + q) \bmod(M)$. Thus energy which should leave through the top end of the spectrum and be lost, shows up in the lower wave numbers, disturbing the accuracy of the calculation in the most important region.

From this point of view, the eigenvalues of the amplification matrix should then, ideally, be unity for low to intermediate wave numbers and drop off rather rapidly in the intermediate- to high-wave-number range.

Figure 1 presents the eigenvalues (squared) of the amplification matrix for four different one-dimensional schemes. For values of α near the stability limit, the proposed method (curves a and a′) has less damping for low wave numbers than does either the Lax–Wendroff Two Step method (curves c and c′) or Leith's scheme (d). It also has less damping as a pure one-dimensional scheme than its

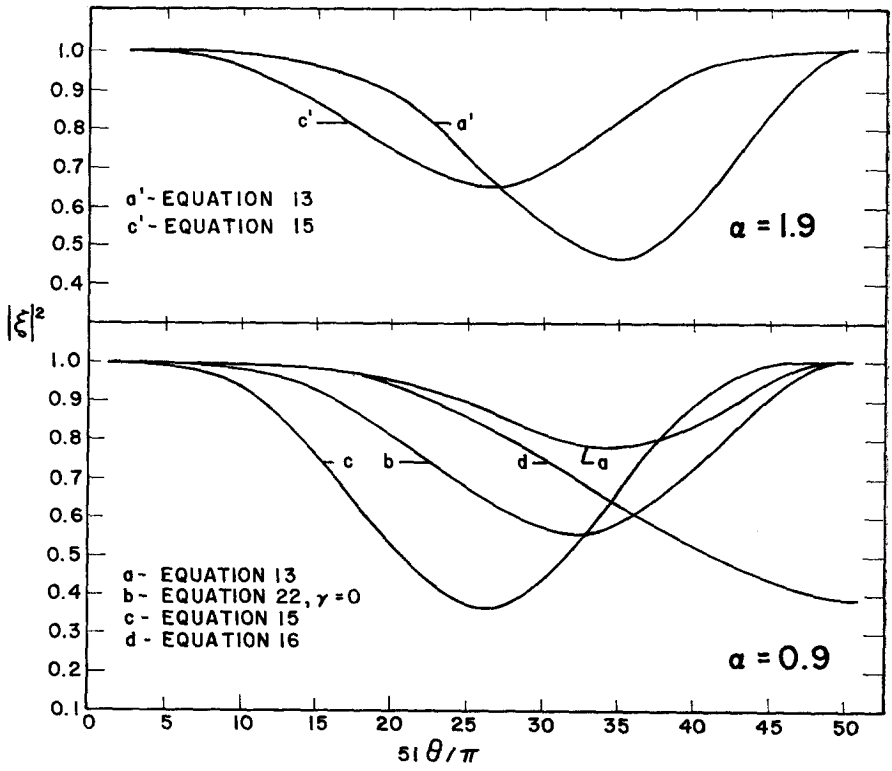

FIG. 1

two-dimensional analog (b) with one velocity component set zero. It should be noted, however, that curve (d) is more desirable from the point of view of aliasing and is almost as good as curve (a) in the low-wave-number range. As α tends toward the stability limit, curve (a′) gains some low-wave-number damping, but is much improved over (a) in the sense of aliasing.

If Burgers' equation is transformed to wave-number space as above, the diffusion term νu_{xx}, gains a coefficient which includes the square of the wave number. Thus the amplitude damping introduced by it increases as the square of the wave number. It is anticipated then that the inclusion of the diffusion term (as, for example, an eddy viscosity) will help to inhibit aliasing.

Phase Error

In addition to amplitude errors, the finite-difference approximations to Eq. (1) also introduce errors in phase. A solution of (1), with $S = 0$, is $\epsilon^{ik(x-ut)}$, so that after a time interval Δt, the correct phase angle is $\delta = -ku\,\Delta t = -\theta\alpha$. Substituting $\psi_j = \epsilon^{ikj\Delta x}$ into (12), the phase angle after a time interval Δt is found to be

$$\sin \delta_1 = -\alpha \sin \theta \left\{1 - \frac{\alpha^2}{2}(1 - \cos\theta)\right\}\left\{1 - \left(1 - \frac{\alpha^2}{4}\right)\left(\frac{\alpha^4}{4}\sin^2\theta\right)(1 - \cos\theta)^2\right\}^{-1/2}$$

The same analysis applied to the Lax–Wendroff Two-Step Scheme and to the Quadratic Advection Scheme [Eq. (16)] give, for the phase angle, respectively,

$$\sin \delta_2 = -\alpha \sin\theta \cos\theta \left\{1 - \alpha^2 \sin^4\theta \left(1 - \frac{\alpha^2}{4}\right)\right\}^{-1/2}$$

and

$$\sin \delta_3 = -\alpha \sin\theta\{1 - \alpha^2(1 - \alpha^2)(1 - \cos\theta)^2\}^{-1/2}.$$

The quantities δ_1/δ, δ_2/δ, and δ_3/δ for selected values of θ and for $\alpha = 0.1$ are presented in Table I. They show that the proposed scheme and the Quadratic Advection Scheme (both second order in space) have approximately the same phase error which is not too large for low wave numbers (small θ) but is quite bad for large wave numbers. The Two-Step Lax–Wendroff Scheme suffers quite a large phase error even at relatively low wave numbers.

Roberts and Weiss [8] have proposed a finite-difference scheme which is second order in time and fourth order in space. In comparing their second-order scheme (second order in time and space) with their "fourth" order scheme, they find the phase error to be substantially reduced by the use of the higher order scheme.

Comparison of the phase error in their second-order scheme with columns one

TABLE I

θ	δ_1/δ	δ_2/δ	δ_3/δ
20°	0.9799	0.9209	0.9800
30°	.9550	.8275	.9554
45°	.9004	.6377	.9015
60°	.8270	.4148	.8291
90°	.6361	0.	.6409
120°	.4125	−0.2074	.4187
150°	.1902	−0.1655	.1944
180°	0.	0.	0.

and three in Table I shows the results to be approximately the same. This indicates then that the phase error in the present scheme can perhaps be decreased by using a higher-order approximation to the spatial derivatives.

Two-Dimensional Advection

In two-dimensional Cartesian coordinates, the advection equation becomes

$$\psi_t + u\psi_x + v\psi_y = 0.$$

ψ is now a function of three independent variables and $\psi(x, y, t) = \psi(k\Delta x, l\Delta y, N\Delta t)$ is denoted by $\psi^N_{k,l}$.

Let the finite difference operators A, B, C, D be defined as follows.

$$A\psi_{k,l} = \frac{\alpha_{k,l}}{4}(\psi_{k+1,l} - \psi_{k-1,l}) - \left(\frac{\alpha_{k,l}}{2}\right)^2(\psi_{k+1,l} - 2\psi_{k,l} + \psi_{k-1,l}), \tag{17}$$

$$B\psi_{k,l} = \frac{\gamma_{k,l}}{4}(\psi_{k,l+1} - \psi_{k,l-1}) - \left(\frac{\gamma_{k,l}}{2}\right)^2(\psi_{k,l+1} - 2\psi_{k,l} + \psi_{k,l-1}), \tag{18}$$

$$C\psi_{k,l} = \frac{\alpha_{k,l}}{2}(\psi_{k+1,l} - \psi_{k-1,l}), \tag{19}$$

$$D\psi_{k,l} = \frac{\gamma_{k,l}}{2}(\psi_{k,l+1} - \psi_{k,l-1}), \tag{20}$$

where $\alpha_{k,l} = u_{k,l}\,\Delta t/\Delta x$ and $\gamma_{k,l} = v_{k,l}\,\Delta t/\Delta y$.

The first step in the cycle is

$$\psi^{N+1/2}_{k,l} = (I - A)(I - B)\,\psi^N_{k,l}$$

which may be thought of as two steps $\psi^* = (I - B)\psi^N$ followed by $\psi^{N+1/2} = (I - A)\psi^*$.

This procedure (known as cascading, or time splitting) is computationally stable so long as the individual steps are stable, but terms are introduced so that the errors are second order rather than third. As in the one-dimensional case, this is acceptible as long as the final step produces errors of third order.

It might be thought that the first step should be $\psi_{k,l}^{N+1/2} = (I - A - B)\psi_{k,l}^N$, but as indicated by Leith [6], this has modes which are unstable, and is therefore unsuitable for stable calculations. It should also be noted that the operator A differs from the one-dimensional operator $\bar{A}$ in that a factor of $\frac{1}{2}$ is missing from the second term. This missing factor is necessary to stabilize the two-dimensional case, and leads to the interesting situation that if $v = 0$ ($\gamma = 0$) the calculation has a condition for stability of $\alpha^2 < 2$ rather than $\alpha^2 < 4$ as occurs in the strictly one-dimensional case.

The final step is

$$\psi_{k,l}^{N+1} = \psi_{k,l}^N - (C + D)\psi_{k,l}^{N+1/2}$$

and the complete step from t to $t + \Delta t$ is thus

$$\psi_{k,l}^{N+1} = (I - (C + D)(I - A)(I - B))\psi_{k,l}^N . \tag{21}$$

As in the one-dimensional case, numerical stability dictates that the eigenvalues of the amplification matrix be on or inside the unit circle. A complete examination of the eigenvalues for all wave numbers and all values of α and γ would be quite complicated and is not done here. The approach taken is to consider the case of equal wave numbers in both x and y directions and to look at some limiting cases. From these, a necessary condition for stability may be derived. It should be borne in mind that the standard stability analysis requires a linearization of the equation, and the final answer as to whether the method is stable or not can with present techniques only be obtained by numerical experimentation.

The eigenvalue of the operator in (21) corresponding to an eigenvector $\epsilon^{ip(k\Delta x + l\Delta y)}$ is

$$\begin{aligned}\xi_p = 1 &- \frac{(\alpha + \gamma)^2 \sin^2\theta}{2}\left\{1 - \frac{\alpha\gamma}{2}(1 - \cos\theta)\right\} \\ &- i(\alpha + \gamma)\sin\theta\left\{1 - (1 - \cos\theta)\left[\frac{\alpha^2 + \gamma^2}{2} + \frac{\alpha\gamma}{4}(1 + \cos\theta)\right.\right. \\ &\left.\left. - \frac{\alpha^2\gamma^2}{4}(1 - \cos\theta)\right]\right\},\end{aligned} \tag{22}$$

where $\theta = p\Delta x = p\Delta y$.

For small wave numbers,

$$\sin\theta = \theta(1 - \tfrac{1}{6}\theta^2) \qquad \text{and} \qquad \cos\theta = (1 - \tfrac{1}{2}\theta^2)$$

Then

$$|\xi_p|^2 = 1 - \tfrac{1}{8}(\alpha + \gamma)^2\,[(\alpha - \gamma)^2 + \alpha^2 + \gamma^2]\,\theta^4 \tag{23}$$

which is stable for all α and γ.

It should be noted, that if the one-dimensional operators are used in the first step of the two-dimensional case; i.e., if $A = \bar{A}$ and B is a similar operator in the v direction, then for the case of θ small, $\alpha = \gamma$, the eigenvalues of the amplification matrix are

$$|\xi|^2 = 1 + \tfrac{1}{2}(\alpha^4\theta^4)$$

which leads to growth and is thus an unusable scheme.

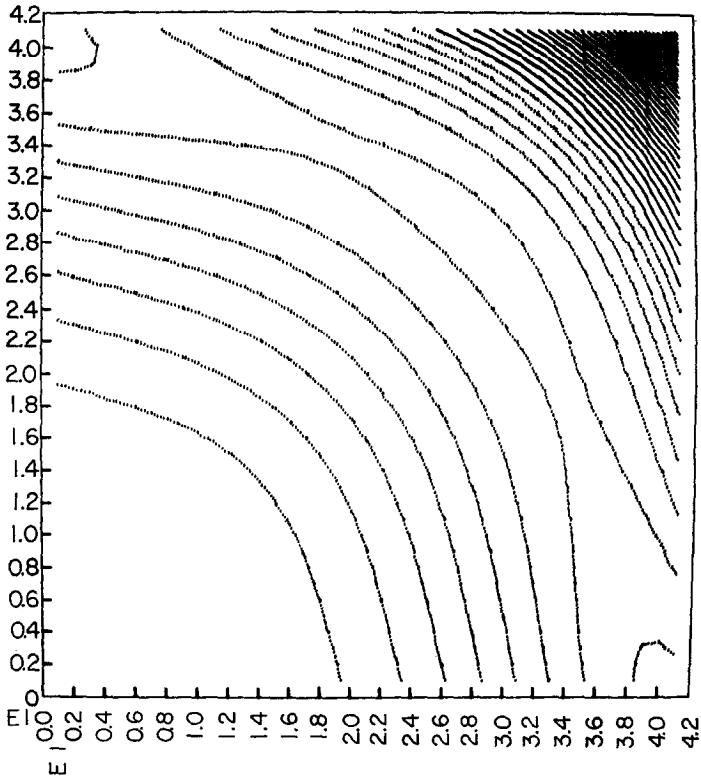

FIG. 2

For large wave numbers $\theta = \pi - \epsilon$ where $0 < \epsilon \ll 1$,

$$\sin\theta = \sin\epsilon \cong \epsilon,$$
$$\cos\theta = -\cos\epsilon \cong -(1 - \tfrac{1}{2}\epsilon^2).$$

Then

$$|\xi_p|^2 = 1 - (\alpha + \gamma)^2\epsilon^2[1 - \alpha\gamma - (1 - \alpha^2)^2(1 - \gamma^2)^2]. \tag{24}$$

For α, γ small, (21) becomes

$$|\xi|^2 \cong 1 - (\alpha + \gamma)^2\epsilon^2\{\tfrac{1}{2}(\alpha - \gamma)^2 + \tfrac{3}{2}(\alpha^2 + \gamma^2)\}, \tag{25}$$

which is obviously stable.

For $\gamma = 0$, (21) becomes

$$|\xi|^2 = 1 - \alpha^2\epsilon^2\{1 - (1 - \alpha^2)^2\}, \tag{26}$$

which requires $\alpha^2 < 2$.

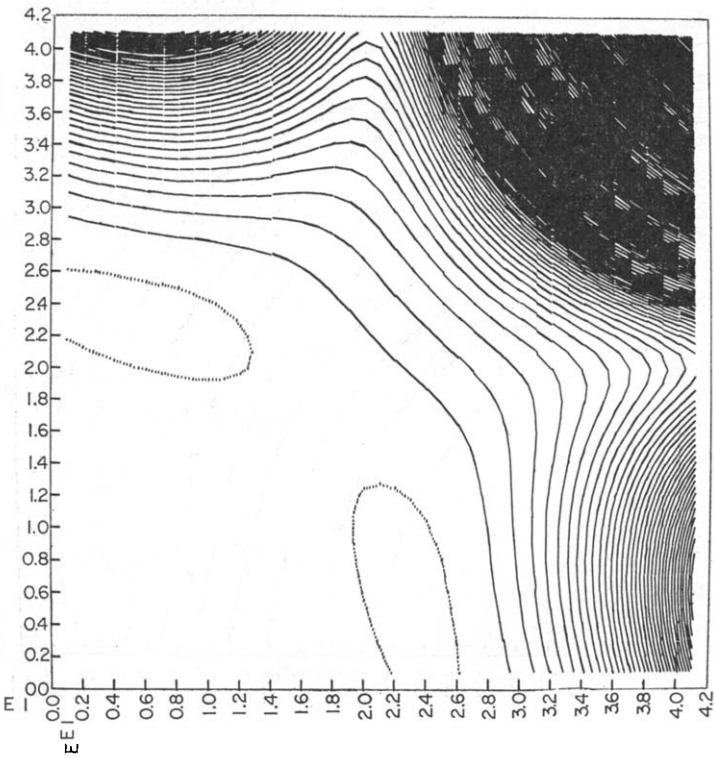

FIG. 3

For $\alpha = \gamma$, (21) becomes

$$| \xi |^2 = 1 - 4\alpha^2\epsilon^2[1 - \alpha^2][1 - (1 - \alpha^2)^3], \tag{27}$$

which requires $\alpha^2 < 1$ hence $\alpha^2 < 1$ and $\gamma^2 < 1$.

For $\gamma = \frac{1}{2}\alpha$,

$$| \xi |^2 = 1 - \tfrac{9}{4}\alpha^2\epsilon^2[1 - \tfrac{1}{2}\alpha^2 - (1 - \alpha^2)^2(1 - \tfrac{1}{4}\alpha^2)^2], \tag{28}$$

which requires at least $\alpha^2 < \frac{8}{5}$.

Figures 2, 3, and 4 present unretouched contour plots of Eq. (22) which were computed by a digital computer and automatically plotted using CRT facilities and available plotting programs. The unnormalized abscissa ($\bar{\alpha}$) and ordinate ($\bar{\gamma}$) are related to α and γ by $\alpha = .05(\bar{\alpha} - 1)$ and $\gamma = .05\ (\bar{\gamma} - 1)$ so that α and γ run from 0 to 2. Broken lines indicate level lines of $| \xi |^2 < 1$ and solid lines indicate level lines of $| \xi |^2 \geqslant 1$. The contour interval is 0.1.

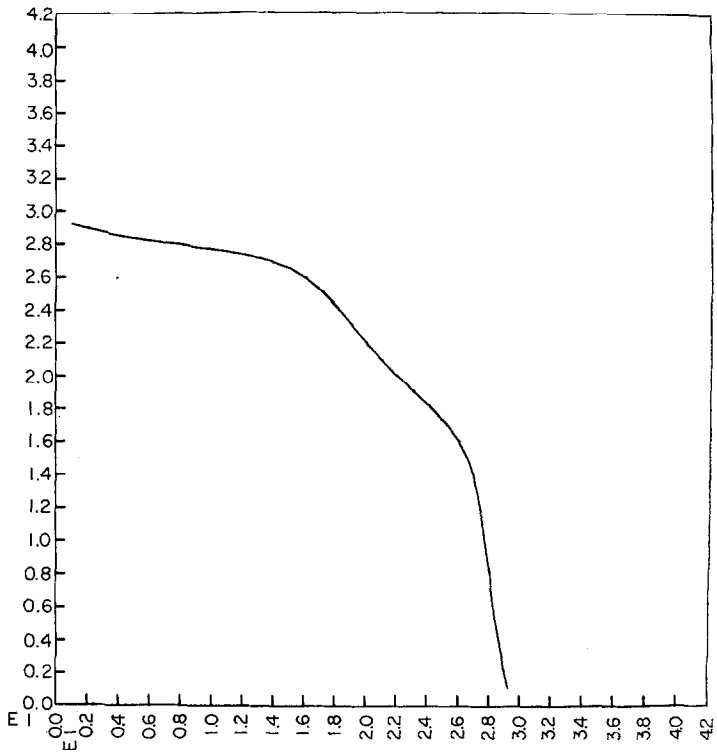

FIG. 4

Figure 2 is for $\theta = .3\pi$; Fig. 3 is for $\theta = .9\pi$ and shows that the stable region decreases as θ increases which is a common experience.

Figure 4 is for $\theta = 0.999\pi$, the lower left-hand corner still containing the stable region, while the upper right band region contains the (very mildly at this wave number) unstable region. From this figure, it would appear that $\alpha^2 + \gamma^2 < 1.8$ will ensure computational stability. This is less than that implied by (28) since the minimum stability apparently is for $0 < \gamma < .4\alpha$.

Numerical Results

Figure 5 shows the results of three numerical experiments performed using Eqs. (10) and (11). For all problems, initial conditions consisted of a Gaussian distribution (curve "a") spread over 36 zones. Periodic spatial boundary conditions were applied so that $\psi(0, t) = \psi(L, t)$, for all time, and in each case u was a constant. The analytic solution then involves only a translation of the initial configuration through a distance $ut = \alpha N \Delta x$.

Results, $\psi(x, t)$, are plotted versus $x - ut$ and in all cases the profile has moved a total distance of 360 zones. Curve "c" is for $\alpha = 0.1$ so that the curve is the

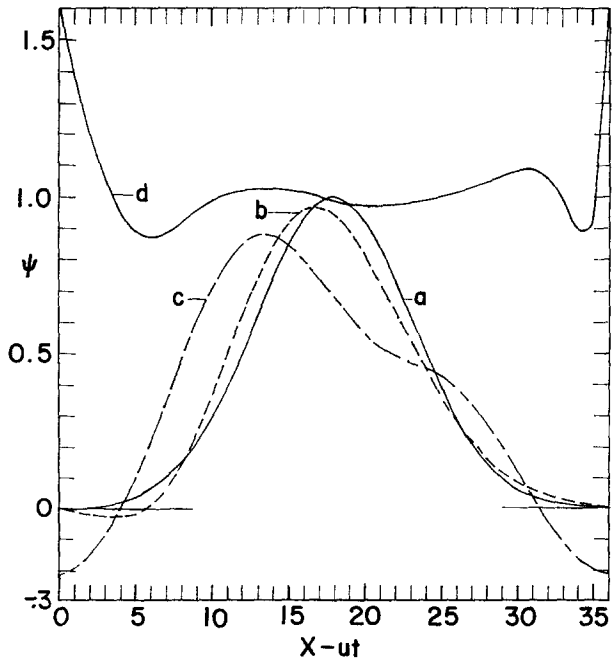

FIG. 5

result of 3600 integration steps. Curve "b" is for $\alpha = 1.8$ and is the result of 200 steps. This shows quite clearly that problems, which are run with Δt near the maximum permissible time increment, profit from increased accuracy as well as from efficient machine-time usage. Unfortunately in most real problems, other phenomena such as the propagation of gravity or sound waves enter into the stability requirements, and inhibit the time step even more than advection.

Fourth-order accuracy in space is obtained by including ψ_{j+2} and ψ_{j-2} in the estimates of the spatial derivatives

$$\bar{A}\psi_j = \frac{\alpha_j}{24}\{8(\psi_{j+1} - \psi_{j-1}) - (\psi_{j+2} - \psi_{j-2})\} - \frac{\alpha_j^2}{48}\{-30\psi_j + 16(\psi_{j+1} + \psi_{j-1}) - (\psi_{j+2} + \psi_{j-2})\}, \tag{29}$$

$$\bar{C}\psi_j = \frac{\alpha_j}{12}\{8(\psi_{j+1} - \psi_{j-1}) - (\psi_{j+2} - \psi_{j-2})\}. \tag{30}$$

A two-step scheme which is second-order accurate in time and fourth order accurate in space is produced by replacing operators (8) and (9) by (29) and (30) and substituting these into (10) and (11). A test problem was run using this scheme, with $\alpha = 0.1$ and 3600 time steps, so that the profile moved 360 zones as in the previous two cases. Results were remarkably improved to the extent that, if plotted on Fig. 5 they would be indistinguishable from curve "a". The ratio of these results to the analytic results is given as curve "d".

It is clear from these test calculations that the fourth-order scheme not only improves the phase error, but it also decreases the amplitude damping substantially.

Conclusions

A finite-difference method has been proposed which is second order in both time and space. The scheme advances the state vector from t to $t + \Delta t$ by a two-stage process which involves less calculations than a similar first-order scheme would involve in covering the same time interval (the first-order scheme being restricted to half the time interval of the proposed scheme). Thus, both increased accuracy and computational speed are achieved at the cost of a slight increase in the complexity of programming.

Comparison of the proposed scheme with centered-time-difference schemes, which compute the same number of mesh points each cycle, indicates that the computer memory requirements for the centered scheme are approximately double those of the proposed scheme. The proposed scheme is also not burdened with the problem of two solutions appearing at adjacent times, which is associated with centered-time-difference schemes.

Every space point is calculated every time cycle so that the problem of two out-of-phase solutions which appear in the Lax–Wendroff scheme can not appear here.

For values of α near the stability limit, the one-dimensional scheme exhibits minimal amplitude damping at low wave numbers which increases to a maximum at about $k = 2\pi/3\Delta x$ and then approaches zero again as k tends to $\pi/\Delta x$. This latter feature of decreasing damping at higher wave numbers is not desirable from the point of view of aliasing, and should be partially eliminated by the inclusion of a diffusive term. The minimal damping at low wave numbers is quite desirable.

Comparison of the phase errors inherent in this scheme with those in a scheme proposed by Roberts and Weiss, which is second order in time and fourth order in space, shows the higher-order scheme in this respect to be more desirable. It is thought then that the proposed scheme can be improved by increasing the order of the spatial truncation error while not changing the order of the time truncation error. Test calculation in one space dimension show this to be true. They also indicate that the higher-order scheme has less amplitude damping associated with it.

Contour plots of the eigenvalues of the amplification matrix are presented for the two-dimensional formulation. These suggest that the inequality $\alpha^2 + \gamma^2 < 1.8$ is necessary for computational stability.

ACKNOWLEDGMENT

This work was done under the auspices of the U. S. Atomic Energy Commission. The proposed scheme was conceived and partially analyzed while the author was at the AEC Computing Center of the Courant Institute of Mathematical Sciences, New York, N. Y.

REFERENCES

1. K. BYRAN, Nonlinear effects of wind-driven ocean circulation, *in* "Methods in Computational Physics (B. Adler, S. Fernbach, and M. Rotenberg, Eds.), Vol. 4, p. 36. Academic Press, New York (1965).
2. P. LAX and B. WENDROFF, *Commun. Pure Appl. Math.* **42**, 381–398 (1964).
3. R. D. RICHTMYER, "A Survey of Difference Methods for Non-Steady Fluid Dynamics," NCAR Technical Note 63-2. National Center for Atmospheric Research, Boulder, Colorado (1963).
4. S. BURSTEIN, *J. Comp. Phys.* **1**, 198–222 (1967).
5. D. HOUGHTON, A. KASAHARA, and W. WASHINGTON, *Monthly Weather Rev.* **94**, 141–150 (1966).
6. C. E. LEITH, Numerical simulation of the earth's atmosphere, *in* (B. Adler, S. Fernbach, and M. Rotenberg, Eds.) Academic Press, New York (1965).
7. N. A. PHILLIPS, An example of non-linear computational instability, *in* "Atmosphere and Sea in Motion," Rossby Memorial Volume (B. Bolin, Ed.). Rockefeller Institute Press, New York (1959).
8. K. V. ROBERTS, and N. O. WEISS, *Math. Computation* **20**, 94 (1966).